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Phase-equivalent potentials from supersymmetry: analytical results for a Natanzon-class potential

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Abstract. Applying the techniques of supersymmetric quantum mechanics we determine closed algebraic expressions for potentials that are phase-equivalent with the generalized Ginocchio potential, which is a member of the rather general Natanzon-potential class. In particular, we discuss the elimination of bound states, the addition of one (or more) bound state at specific energies and also mention transformations that leave the spectrum unchanged. Our work represents the application of the abstract mathematical formalism developed recently for the modification of the spectrum of potentials without changing the S-matrix and the phase shifts. A new aspect of our work is that in addition to the new potential function, we give closed analytical expressions for the transformed Jost functions and bound-state wavefunctions. Furthermore, this work is the first example for generating phase-equivalent partners of a potential outside the relatively simple shape-invariant potential class.

1. Introduction

Phase-equivalent potentials appear naturally in various branches of physics. Besides having the same phase shifts, these potentials might support a different number of bound states if (at least) one of them is singular at the origin. The formalism of supersymmetric quantum mechanics (SUSYQM) was found to be especially suitable for describing phase-equivalent potentials.

Since its introduction SUSYQM has evolved into a highly sophisticated method of handling isospectral quantum mechanical systems. (For recent reviews on SUSYQM see [1, 2].) Its early applications concerned mainly single transformations by which the ground state of a potential could be removed or a new ground state could be introduced, depending on the solution of the Schrödinger equation used in the process. It was noticed that these manipulations also change the r^{-2}-like singularity of the potentials and modify the phase shifts in a characteristic way [3, 4]. Later it was shown [5] that by using pairs of such transformations one can construct potentials that lead to the same phase shifts as the original potential despite the different numbers of bound states the two potentials support, and this result was interpreted in terms of the generalized Levinson theorem [6]. This aspect of SUSYQM also allowed straightforward interpretation of the long standing problem represented by the duality of 'deep'- and 'shallow'-type potentials used in the description of interacting composite particles. The relation of SUSYQM to other methods of analysing isospectral potentials, such as the inverse scattering theory [7] has also been discussed [3, 4, 8, 9].

More recently the formalism of generating phase-equivalent potentials has been developed to a stage where, in principle, arbitrary modifications of the energy spectrum are possible [10, 8, 11]. The final potential and the wavefunctions are expressed in terms of compact formulae depending on integrals and determinants composed of physical and unphysical solutions of the Schrödinger equation. These expressions can be evaluated by numerical techniques in general.

The generality and compactness of the formulae also raises the question whether it is possible to find examples where the whole procedure can be performed in an analytical way, i.e. whether there are cases where the resulting potential is obtained in a closed algebraic expression. Such investigations are also motivated by the renewed interest in exactly solvable quantum mechanical problems also raised partly by SUSYQM (see for example [12, 1, 2] and references therein). Efforts in this direction have been limited to some particular examples from the well known shape-invariant potential class [13]. The ground state of the Coulomb [14, 15], Morse and Hulthén [16] potentials have been removed, and somewhat more general transformations have been formulated for the Coulomb [15, 10] and the generalized Pöschl–Teller [17] potentials. Other potentials have also been studied without analysing the effect of the transformations on their spectra [18]. Apart from their aesthetic value, the importance of fully analytical transformations lies in the fact that exact results can be obtained even in critical conditions when the numerical techniques might not be safely controlled. Handling complex potentials can raise such problems, for example [19, 20].

The abstract formalism developed for the derivation of phase-equivalent partners of known potentials can be applied to the rather general Natanzon potential class [21], which contains all the shape invariant potentials [13] as special cases. In order to demonstrate this we derive potentials which are phase-equivalent with the generalized Ginocchio potential [22], which is probably the most well known member of the Natanzon potential class.

In section 2 we briefly review the basic transformations used in SUSYQM. We study the generalized Ginocchio potential in section 3 and formulate analytical transformations for removing or adding bound states in the spectrum, and introducing new parameters in the potentials while leaving the spectrum unchanged. Finally, in section 4 we summarize the results and give a brief outlook for further studies.

2. The basic transformations of SUSYQM

Let us consider the radial Schrödinger equation (with $\hbar^2/2m = 1$)

$$H_0\varphi_0(k,r) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_0(r)\right)\varphi_0(k,r) = k^2\varphi_0(k,r) \tag{1}$$

and the factorization of the corresponding Hamiltonian

$$H_0 = A_0^+ A_0^- + E_0 \tag{2}$$

where the factorization energy $E_0 = k_0^2$ does not exceed the ground-state energy $E_0^{(0)}$, and

$$A_0^\pm = \pm\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\varphi_0'(k_0,r)}{\varphi_0(k_0,r)}. \tag{3}$$

The lower index 0 of H_0, V_0, φ_0 and $A_0^\pm$ labels the original Hamiltonian, etc. as opposed to the corresponding quantities related to further problems derived later on. Due to $E_0 \leqslant E_0^{(0)}$

Table 1. Properties of the three transformations resulting in potentials phase-equivalent with the original potential in equation (1). V_0 is supposed to be singular at the origin as $V_0(r) \simeq \nu(\nu+1)r^{-2}$, which accounts for the centrifugal term also.

Transformation	Removes a bound state	Adds a bound state ($\nu > 1$ only)	Unchanged spectrum
Solution φ_0	$\psi_0^{(i)}$	f_0	$\psi_0^{(i)}$
Parameter β	-1	$\alpha > 0$	$\alpha/(1-\alpha)$, $\alpha > 0$
Fact. energy E_0	$E_0^{(i)} < 0$	$E_0 \neq E_0^{(i)}$, $E_0 < 0$	$E_0^{(i)} < 0$
$\lim_{r\to 0}\varphi_0$	$r^{\nu+1}$	$r^{-\nu}$	$r^{\nu+1}$
$\lim_{r\to\infty}\varphi_0$	$\exp(-\|k_0^{(i)}\|r)$	$\exp(-\|k_0\|r)$	$\exp(-\|k_0^{(i)}\|r)$
Singularity of V_2	$(\nu+2)(\nu+3)r^{-2}$	$(\nu-2)(\nu-1)r^{-2}$	$\nu(\nu+1)r^{-2}$
$F_2(k)/F_0(k)$	$k^2/(k^2+\|k_0^{(i)}\|^2)$	$(k^2+\|k_0\|^2)/k^2$	1

the solutions of (1) are nodeless. The supersymmetric partner of H_0 is defined as

$$H_1 \equiv -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_1(r) = A_0^- A_0^+ + E_0 = H_0 - 2\frac{\mathrm{d}}{\mathrm{d}r}\frac{\varphi_0'(k_0,r)}{\varphi_0(k_0,r)}. \tag{4}$$

The properties of $V_0(r)$ and $V_1(r)$ are connected in a characteristic way determined by the nature of the solution φ_0 [4, 10]. In the following we call the solution φ_0 physical if it is a bound-state wavefunction or a scattering solution of V_0, and refer to it as an unphysical solution in any other case.

Further potentials can be derived by combining single SUSYQM transformations. Pairs of such transformations can be employed to generate potentials phase-equivalent with the original one provided that the factorization energies are chosen to be equal, guaranteeing that the original scattering phases are restored after the second step. For the resulting transformation, the factorization energy is no longer required to be smaller than $E_0^{(0)}$. As described in [10], for example, only three non-trivial combinations are possible, and the resulting potential is written as

$$V_2(r) = V_0(r) + 2\frac{\mathrm{d}}{\mathrm{d}r}\frac{(\varphi_0(k_0,r))^2}{\beta + \int_r^\infty (\varphi_0(k_0,t))^2\mathrm{d}t}. \tag{5}$$

The appropriate choices of $\varphi_0(k_0,r)$ and β are summarized in table 1, where the properties of the three basic transformation types are also given. In table 1, $\psi_0^{(i)}$ represents the wavefunction of an arbitrary bound state at energy $E_0^{(i)} = k_0^{(i)2}$ while f_0 represents a solution decreasing at infinity and singular at the origin, at any negative energy $E_0 = k_0^2$ where there is no bound state. The integral in the denominator of (5) always converges because the chosen $\varphi_0(k_0,r)$ decrease exponentially at large r, in all cases. The wavefunctions of $V_2(r)$ are expressed in terms of the original wavefunctions $\varphi_0(k,r)$ and the (physical or unphysical) factorization functions $\varphi_0(k_0,r)$ as

$$\varphi_2(k,r) = \mathcal{N}^{-\frac{1}{2}}\left(\varphi_0(k,r) - \varphi_0(k_0,r)\frac{\int_r^\infty \varphi_0(k_0,t)\varphi_0(k,t)\,\mathrm{d}t}{\beta + \int_r^\infty (\varphi_0(k_0,t))^2\,\mathrm{d}t}\right). \tag{6}$$

Here $\mathcal{N} = 1$, except for $k = k_0$ when $\varphi_2(k_0,r)$ is physical, in which case $\mathcal{N} = \alpha$ [11].

The ratio $F_2(k)/F_0(k)$ of the original and the transformed Jost functions (see the last line of table 1 and also [23]) confirms that the S-matrix is unchanged: $S_2(k) = S_0(k)$. In the case of the removal of a bound state, the zero of the initial Jost function at $\mathrm{i}|k_0^{(i)}|$ is suppressed, and a pole is added at $-\mathrm{i}|k_0^{(i)}|$ in order to restore the S-matrix. In the same way, in the case of the addition of a bound state, two zeros are added simultaneously to the Jost function: one at $\mathrm{i}|k_0|$ which corresponds to the new bound state, and one at $-\mathrm{i}|k_0|$ which restores the S-matrix. Finally, the Jost function is unchanged when the spectrum is not modified. Phase equivalence can also be verified directly on the asymptotic behaviour of (6). Note that the third possibility also contains implicitly the trivial $V_2(r) = V_0(r)$ transformation for $\alpha \to 1$, and also the removal of a bound state in the $\alpha \to \infty$ limit.

Further potentials phase-equivalent with $V_0(r)$ can be derived by iterating transformation pairs [8, 11]. The equivalent of equation (5) for multiple spectrum modifications can be written as a compact formula involving a determinant containing integrals of physical and unphysical solutions satisfying (1) [8, 11],

$$V_{2m}(r) = V_0(r) - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2} \ln\det\left(\beta_i\delta_{ij} + \int_r^\infty \varphi_0(k_i, t)\varphi_0(k_j, t)\,\mathrm{d}t\right) \tag{7}$$

where the k_i correspond to m different factorization energies $E_i = k_i^2$. A similar formula is available for a generalization of (6) with equations (20) and (21) of [11]. These transformations are, in principle, capable of generating arbitrary modifications of the energy spectrum while keeping the scattering phases unchanged. The determinant form also suggests that the final result is independent of the sequence of the individual transformation pairs, because exchanging two of them merely corresponds to exchanging two columns of the determinants.

3. Phase-equivalent partners of the generalized Ginocchio potential

3.1. Properties of the generalized Ginocchio potential

The first version of the Ginocchio potential was introduced as a one-dimensional quantum mechanical problem which is symmetric with respect to the $x \to -x$ transformation [24]. Later it was generalized to a radial problem [22], which also contains an r^{-2}-like singular term at the origin and allows a particular functional form of an effective mass as well. This effective (i.e. coordinate-dependent) mass is clearly incompatible with the simple Schrödinger equation in (1), therefore here we consider a special case allowing constant mass. In what follows we define the generalized Ginocchio potential as

$$\begin{aligned}V_0(r) = -\frac{\gamma^4}{\gamma^2+\sinh^2 u}\bigg[& s(s+1)+1-\gamma^2-\frac{5\gamma^2(1-\gamma^2)^2}{4(\gamma^2+\sinh^2 u)^2} \\ & -\frac{3(1-\gamma^2)(3\gamma^2-1)}{4(\gamma^2+\sinh^2 u)} - \lambda(\lambda-1)\coth^2 u\bigg]\end{aligned} \tag{8}$$

where we changed the notation of [22] to make it more suitable for our purposes. This form can be obtained from the original formulae by setting $a = 0$, $\alpha_l = \lambda - \frac{1}{2}$, $\nu_l = s$, $\beta_{nl} = \mu$, $\lambda = \gamma$ and $y = \sinh u(\gamma^2 + \sinh^2 u)^{-\frac{1}{2}}$.

The (generalized) Ginocchio potential is an example for 'implicit' potentials, because it is expressed in terms of a function $u(r)$ which is known only in the implicit $r(u)$ form:

$$\begin{aligned}r = \frac{1}{\gamma^2}\big[& \tanh^{-1}((\gamma^2+\sinh^2 u)^{-\frac{1}{2}}\sinh u) \\ & +(\gamma^2-1)^{\frac{1}{2}}\tan^{-1}((\gamma^2-1)^{\frac{1}{2}}(\gamma^2+\sinh^2 u)^{-\frac{1}{2}}\sinh u)\big].\end{aligned} \tag{9}$$

r can take values from the positive half axis, which is mapped by the monotonously increasing implicit $u(r)$ function onto itself. This function is actually the solution of an ordinary first-order differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\gamma^2 \cosh u}{(\gamma^2 + \sinh^2 u)^{\frac{1}{2}}} \tag{10}$$

defining a variable transformation connecting the Schrödinger equation with the differential equation [25] of the Jacobi (and Gegenbauer) polynomials [26, 27]. It can be seen from equations (9) and (10) that $u(r)$ behaves approximately as γr near the origin, and as $\gamma^2 r$ for large values of r. In the $\gamma \to 1$ limit u becomes identical with r, and (8) reduces to the generalized Pöschl–Teller potential.

Bound states are located at

$$E_n = -\gamma^4 \mu_n^2 \tag{11}$$

where n varies from 0 to $n_{\max}$ defined below and

$$\mu_n = \frac{1}{\gamma^2}\left[-\left(2n + \lambda + \frac{1}{2}\right) + \left[\left(2n + \lambda + \frac{1}{2}\right)^2 (1 - \gamma^2) + \gamma^2 \left(s + \frac{1}{2}\right)^2\right]^{\frac{1}{2}}\right]. \tag{12}$$

All the terms in (8) are finite at the origin, with the exception of the last one, which shows r^{-2}-like singularity there, and can be considered either as an approximation of the centrifugal term with $l = \lambda - 1$ (λ integer), or as a part of a singular potential with arbitrary $l \neq \lambda - 1$. Setting $\lambda = 1$ we get the 'simple' Ginocchio potential [24] defined on the line. In what follows we assume that $\lambda \geqslant 1$ holds.

The bound-state wavefunctions are expressed in terms of Jacobi polynomials

$$\psi_0^{(n)}(r) = \mathcal{N}_n (\gamma^2 + \sinh^2 u)^{\frac{1}{4}} (\sinh u)^{\lambda} (\cosh u)^{-\mu_n - \lambda - \frac{1}{2}} P_n^{(\mu_n, \lambda - \frac{1}{2})}(2\tanh^2 u - 1) \tag{13}$$

which reduce to Gegenbauer polynomials [25] for $\lambda = 1$. The normalization is given by

$$\mathcal{N}_n = \left[\frac{2\gamma^2 n!\ \Gamma(\mu_n + \lambda + n + \frac{1}{2})\mu_n(\mu_n + \lambda + 2n + \frac{1}{2})}{\Gamma(\mu_n + n + 1)\Gamma(\lambda + n + \frac{1}{2})(\mu_n\gamma^2 + \lambda + 2n + \frac{1}{2})}\right]^{\frac{1}{2}}. \tag{14}$$

Considering that the $r \to \infty$ asymptotical limit corresponds to $u \to \infty$ (see equation (9)), the wavefunctions become zero asymptotically if $\mu_n > 0$ holds. Applying this condition to equation (12) we find that the number of bound states is set by $n_{\max} < \frac{1}{2}(s - \lambda)$. To illustrate phase-equivalent transformations we use the $l = 0$ reference potential with $s = 8$, $\lambda = 3.25$ and $\gamma = 15$. This potential and its three bound-state wavefunctions are plotted by dotted lines in figures 1–3.

Later on we shall use the Jost solutions of equation (1) with potential (8) satisfying [28]

$$f_0^{\mathrm{Jost}}(k, r) \to_{r\to\infty} \mathrm{i}^{\lambda - 1} \exp(\mathrm{i}kr). \tag{15}$$

They can be expressed in terms of two linearly independent solutions for arbitrary energy $E = k^2$ and can be written as

$$\begin{aligned} f_0^{\mathrm{Jost}}(k, r) = {} & \exp(\mathrm{i}kr_1)(\gamma^2 + \sinh^2 u)^{\frac{1}{4}} (-\mathrm{i}\sinh u)^{1-\lambda} (\cosh u)^{-\mu(k) + \lambda - \frac{3}{2}} \\ & \times F\left(\frac{1}{2}(\mu(k) - \lambda + \sigma(k) + 2),\right. \\ & \left.\frac{1}{2}(\mu(k) - \lambda - \sigma(k) + 1); \mu(k) + 1; \frac{1}{\cosh^2 u}\right) \end{aligned} \tag{16}$$

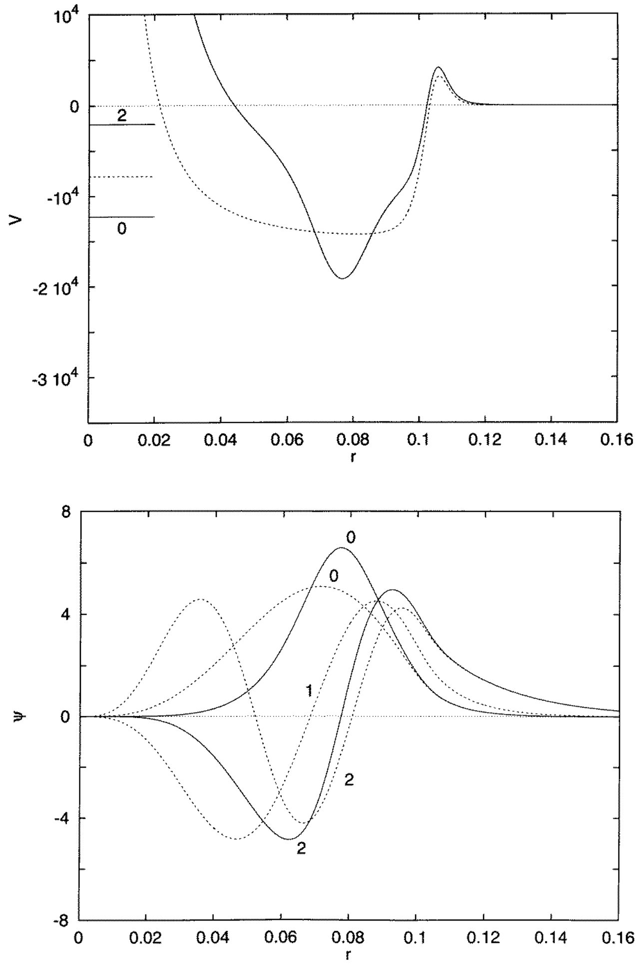

Figure 1. Potential $V_2(r)$ of equation (29), obtained by the removal of the first excited state of the reference potential $V_0(r)$ (upper panel), and the wavefunctions of the two remaining bound states (lower panel). The reference potential (as in equation (8) with $s = 8$, $\lambda = 3.25$ and $\gamma = 15$) and its bound-state wavefunctions are represented by broken curves.

where

$$\sigma(k) = -\tfrac{1}{2} + [\mu^2(k)(1-\gamma^2) + (s+\tfrac{1}{2})^2]^{\frac{1}{2}} \tag{17}$$

$$\mu(k) = -\frac{\mathrm{i}k}{\gamma^2} \tag{18}$$

and

$$r_1 = \frac{1}{\gamma^2}\left[(\gamma^2-1)^{\frac{1}{2}}\tan^{-1}(\gamma^2-1)^{\frac{1}{2}} - \ln\left(\frac{\gamma}{2}\right)\right] \tag{19}$$

as in [22]. The Jost solutions allow expressing the Jost function as

$$F_0(k) = \left(-\frac{k}{2}\right)^{\lambda-1}\frac{\pi^{\frac{1}{2}}}{\Gamma(\lambda-\frac{1}{2})}\lim_{r\to 0}(r^{\lambda-1}f_0^{\mathrm{Jost}}(k,r))$$

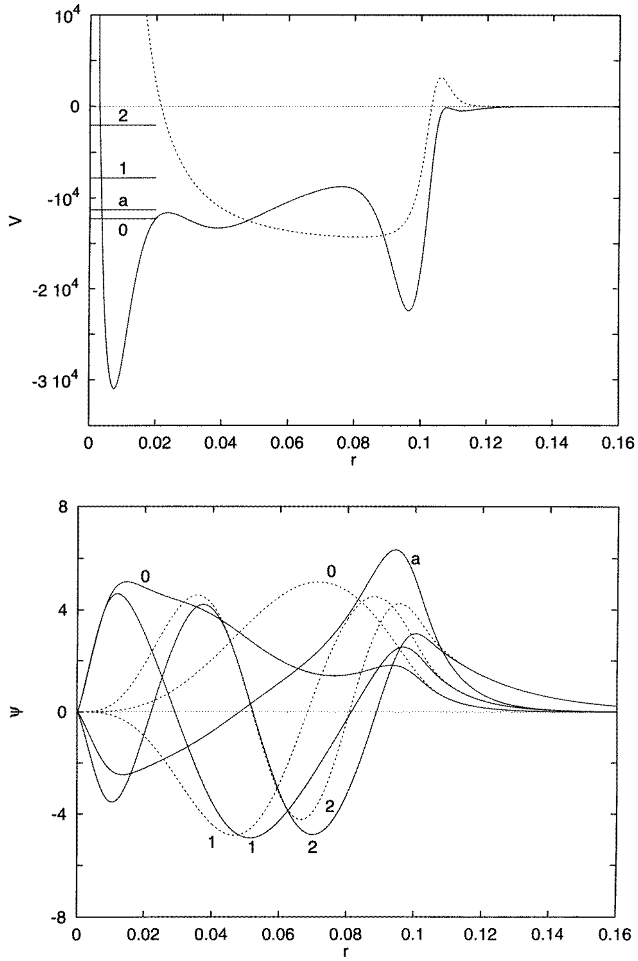

Figure 2. Potential $V_2(r)$ of equation (41) obtained by adding a new bound state (a) between the two lowest states of the reference potential $V_0(r)$ at $E = -11\,289.063$ (upper panel), and the wavefunctions of the four states (lower panel). Here we considered $M = 3$ and set the new parameter as $\alpha_3 = 0.005$. (See also the caption of figure 1.)

$$= \left(-\frac{\mathrm{i}k}{2}\right)^{\lambda-1} \frac{\pi^{\frac{1}{2}}\gamma^{-\lambda+\frac{3}{2}} \exp(\mathrm{i}kr_1)\Gamma(1+\mu(k))}{\Gamma(\frac{1}{2}(\mu(k)+\lambda-\sigma(k)))\Gamma(\frac{1}{2}(\mu(k)+\lambda+\sigma(k)+1))}. \tag{20}$$

This provides the S-matrix as

$$S_0(k) = \exp(2\mathrm{i}\delta_0(k)) = (-1)^{\lambda-1}\frac{F_0(-k)}{F_0(k)} \tag{21}$$

which, together with the substitutions discussed previously, gives the result of [22], up to a $(-1)^l$ phase. This difference is due to the fact that, for the definition of this S-matrix, we consider here that the Ginocchio potential is a singular potential in the $l = 0$ partial wave, rather than a regular potential in the $l = \lambda - 1$ partial wave. This is quite natural since we allow here λ to be noninteger. This convention also explains the unusual factors appearing in equation (15), the first line of equation (20), and in the definition of the S-matrix in terms of the Jost function in equation (21), as explained in [28]. As can be verified by

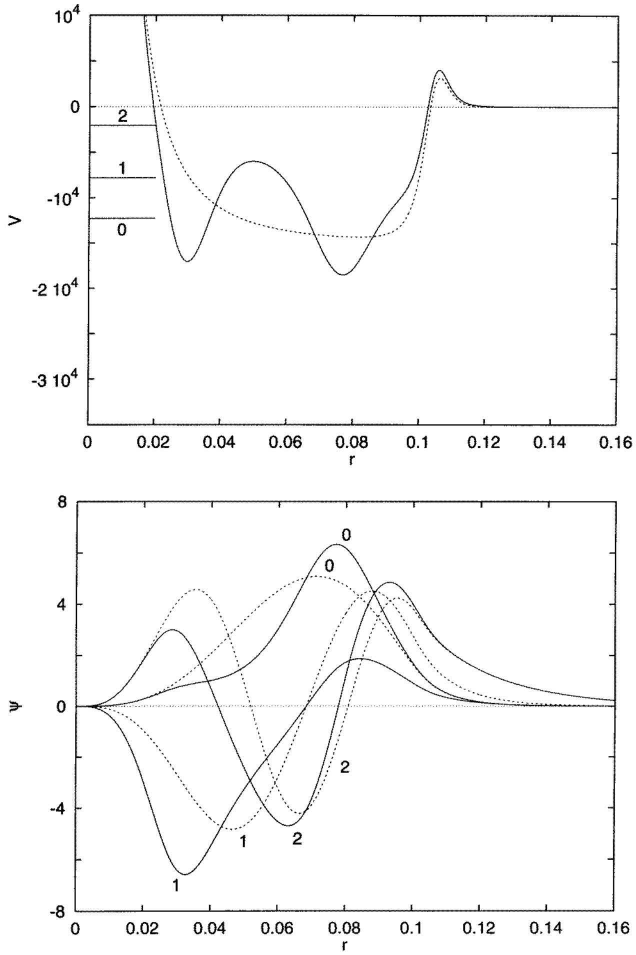

Figure 3. Potential of the type (48) obtained by modifying the normalization constant of the first excited state of the reference potential $V_0(r)$ (upper panel) and the corresponding wavefunctions (lower panel). We have used $\alpha = 10$. (See also the caption of figure 1.)

equation (21), the S-matrix tends to 1 for $k \to 0$, and to $\exp[\mathrm{i}\pi(1-\lambda)]$ at infinity. This is in accordance with the Levinson theorem, generalized for singular potentials [6],

$$\delta_0(0) - \delta_0(\infty) = \left(n_{\max} + 1 + \frac{\lambda - 1}{2}\right)\pi. \qquad (22)$$

3.2. Removal of bound states

Let us assume that we want to eliminate the bound state with quantum number N. Following the notation of equation (13) the Nth wavefunction can be written in a polynomial form

$$\psi_0^{(N)}(r) = (\gamma^2 + \sinh^2 u)^{\frac{1}{4}} (\sinh u)^{\lambda} (\cosh u)^{-\mu_N - 2N - \lambda - \frac{1}{2}} p_N(\cosh u) \qquad (23)$$

where the coefficients of

$$p_N(\cosh u) = \mathcal{N}_N (\cosh u)^{2N} P_N^{(\mu_n, \lambda - \frac{1}{2})}(2\tanh^2 u - 1) \equiv \sum_{j=0}^{N} c_j^{(N)} (\cosh u)^{2j} \qquad (24)$$

are written as

$$c_j^{(N)} = \mathcal{N}_N(-1)^{N-j}\frac{\Gamma(\mu_N+N+1)\Gamma(\mu_N+2N+\lambda+\frac{1}{2}-j)}{j!(N-j)!\Gamma(\mu_N+N+1-j)\Gamma(\mu_N+N+\lambda+\frac{1}{2})}. \tag{25}$$

The same formulae hold, of course, for any other bound-state wavefunction, which we label with quantum number n.

In order to derive the new potential and the new bound-state wavefunctions using (5) and (6), the substitutions $\varphi_0(k_0,r)=\psi_0^{(N)}(r)$ and $\varphi_0(k,r)=\psi_0^{(n)}(r)$ have to be made now along with $\beta=-1$ in table 1. The integrals appearing in (5) and (6) can then be expressed in terms of the general formula

$$I_{Nn}(r)=\int_0^r \psi_0^{(N)}(t)\psi_0^{(n)}(t)\,\mathrm{d}t=\frac{(\sinh u)^{2\lambda+1}}{(\cosh u)^{\mu_N+\mu_n+2N+2n+2\lambda-1}}G_{Nn}(u) \tag{26}$$

where $G_{Nn}(u)$ is defined as

$$\begin{aligned}G_{Nn}(u)=\frac{1}{(2\lambda+1)\gamma^2}\sum_{m=0}^{n+N}&\frac{d_m^{(Nn)}(\cosh u)^{2m}}{\mu_N+\mu_n+2N+2n+2\lambda+1-2m}\\&\times\left[\frac{(2\lambda+1)(\gamma^2-1)}{\cosh^2 u}+((\mu_N+\mu_n+2N+2n-2m)\gamma^2+2\lambda+1)\right.\\&\left.\times F(-\tfrac{1}{2}(\mu_N+\mu_n)-N-n+m+1,1;\lambda+\tfrac{3}{2};-\sinh^2 u)\right].\end{aligned} \tag{27}$$

This expression can be derived using [29, equations 3.194.1 and 2] after rearranging the summation for the two running indices appearing in the polynomial form of $\psi_0^{(N)}(r)$ and $\psi_0^{(n)}(r)$. This also requires the introduction of the coefficients

$$d_m^{(Nn)}\equiv\sum_{j=\max(0,m-n)}^{\min(m,N)}c_j^{(N)}c_{m-j}^{(n)}. \tag{28}$$

The resulting potential which has bound states at E_n in (11), except for $n=N$ takes the form

$$\begin{aligned}V_2(r)=V_0(r)+2\frac{\gamma^2+\sinh^2 u}{\cosh^4 u\sinh^2 u}&\left[\frac{(p_N(\cosh u))^2}{G_{NN}(u)}\right]^2-\frac{2\gamma^2(p_N(\cosh u))^2}{G_{NN}(u)}\\&\times\left[\frac{1}{\gamma^2+\sinh^2 u}-\frac{2\mu_N+4N+2\lambda+1}{\cosh^2 u}+\frac{2\lambda}{\sinh^2 u}+\frac{2}{\cosh u}\frac{p_N'(\cosh u)}{p_N(\cosh u)}\right]\end{aligned} \tag{29}$$

while the new bound-state wavefunctions are

$$\begin{aligned}\psi_2^{(n)}(r)=(\gamma^2+\sinh^2 u)^{\frac{1}{4}}&(\sinh u)^{\lambda}(\cosh u)^{-\mu_n-2n-\lambda-\frac{1}{2}}\\&\times\left[p_n(\cosh u)-p_N(\cosh u)\frac{G_{Nn}(u)}{G_{NN}(u)}\right].\end{aligned} \tag{30}$$

We note that in the $\gamma\to 1$ limit equations (26) to (29) reduce to the corresponding formulae derived for the generalized Pöschl–Teller potential [17].

Figure 1 shows $V_2(r)$ (as in (29)) obtained by removing the first excited state ($N=1$) of the reference potential (plotted by dotted line). In accordance with the generalized Levinson theorem (22) the $(\lambda-1)\lambda r^{-2}$-type singularity of $V_0(r)$ has changed to $(\lambda+1)(\lambda+2)r^{-2}$ for $V_2(r)$, formally increasing the value of λ with two units. The corresponding wavefunctions are also presented in figure 1 with the corresponding initial wavefunctions as broken curves.

The Jost function of V_2 is directly related to that of V_0 (see table 1), which is analytically known (equation (20)). The S-matrices of V_0 and V_2 are thus identical.

With equation (7), m arbitrary bound states can be removed. For simplicity, we focus on the $m = 2$ case but more general potentials can be obtained for arbitrary m in an obvious way. After removal of the bound states at energies E_{N_1} and E_{N_2}, the potential reads

$$V_4(r) = V_0(r) - 2\frac{\mathrm{d}}{\mathrm{d}r}\frac{(\psi_0^{(N_1)})^2 I_{N_2N_2} + (\psi_0^{(N_2)})^2 I_{N_1N_1} - 2\psi_0^{(N_1)}\psi_0^{(N_2)} I_{N_1N_2}}{I_{N_1N_1}I_{N_2N_2} - (I_{N_1N_2})^2} \tag{31}$$

where all functions depend on r which is implied. Note that due to equation (26) the derivative of $I_{N_iN_j}$ is simply the product of $\psi_0^{(N_i)}$ and $\psi_0^{(N_j)}$. The wavefunctions of the remaining bound states at energies E_n can also be written in terms of the same objects as third-order determinants [8, 11].

3.3. Addition of new bound states at specific energies

In order to add a new bound state at energy $E = k^2$ one has to apply (5) with $\varphi_0(k, r)$ chosen as an f-type solution of (1) which is irregular at the origin (with $\lambda \geqslant 3$ at least) and exponentially goes to zero asymptotically. Now β in table 1 remains arbitrary, representing an additional parameter (α) of the resulting potential. The required unphysical solution of the Schrödinger equation is proportional to the Jost function (16) for $\mathrm{Im}(k) > 0$; we choose here the phase as

$$f_0(k, r) = \mathrm{i}^{1-\lambda}\exp(-\mathrm{i}kr_1) f_0^{\mathrm{Jost}}(k, r). \tag{32}$$

Substituting this function in equations (5) and (6) integrals containing squared hypergeometric functions would have to be determined, which might not be expressed in closed form in general. Although integration by parts can bring these formulae to somewhat simpler expressions, their evaluation becomes relatively straightforward only when the hypergeometric functions $F(a, b; c; \cosh^{-2} u)$ reduce to polynomial form, i.e. when either a or b is a nonpositive integer. This, of course, reduces the generality of the example considered here, and this is manifested in the fact that new bound states can be inserted only at specific energies.

From the $b = -M$ choice we get $-\frac{\mathrm{i}k}{\gamma^2} - \lambda + 1 - \sigma(k) + 2M = 0$, which means that a new bound state can be introduced at the negative energies

$$E_M = k_M^2 \equiv -\gamma^4\tau_M^2 \tag{33}$$

where

$$\tau_M = \frac{1}{\gamma^2}\left[-\left(2M - \lambda + \frac{3}{2}\right) + \left[\left(2M - \lambda + \frac{3}{2}\right)^2(1-\gamma^2) + \gamma^2\left(s + \frac{1}{2}\right)^2\right]^{\frac{1}{2}}\right]. \tag{34}$$

From $\mathrm{i}k < 0$ it follows that conditions for the allowed values of M (i.e. the possible new energy eigenvalues) are determined by $M < \frac{1}{2}(s - \lambda) + \lambda - \frac{1}{2}$, where $\frac{1}{2}(s - \lambda)$ represents the upper limit for the number of bound states the original potential can support. This indicates that a more singular generalized Ginocchio potential (8) (i.e. one with larger λ) can accommodate the new bound state in more possible positions if the other parameter s is fixed. This remains true when we set $s - \lambda$, i.e. the number of bound states fixed.

Considering the special case discussed above the f-type solution takes the polynomial form

$$f_0^{(M)}(r) = (\gamma^2 + \sinh^2 u)^{\frac{1}{4}}(\sinh u)^{1-\lambda}(\cosh u)^{-\tau_M - 2M + \lambda - \frac{3}{2}} q_M(\cosh u) \tag{35}$$

where

$$q_M(\cosh u) \equiv \sum_{j=0}^{M} b_j^{(M)} (\cosh u)^{2j} \tag{36}$$

and

$$b_j^{(M)} = (-1)^{M-j} \frac{M!\Gamma(\tau_M + 2M - \lambda + \frac{3}{2} - j)\Gamma(\tau_M + 1)}{j!(M-j)!\Gamma(\tau_M + M + 1 - j)\Gamma(\tau_M + M - \lambda + \frac{3}{2})}. \tag{37}$$

With this notation the integral in (5) can be expressed in terms of the implicit $u(r)$ function (9). For the sake of generality we present here the integral of two f-type solutions (35) belonging to M and M':

$$\begin{aligned} J_{MM'}(r, \alpha_M) &= \delta_{MM'}\alpha_M + \int_r^\infty f_0^{(M)}(t) f_0^{(M')}(t)\,\mathrm{d}t \\ &= \frac{(\sinh u)^{3-2\lambda}}{(\cosh u)^{\tau_M + \tau_{M'} + 2M + 2M' - 2\lambda + 3}} H_{MM'}(u, \alpha_M) \end{aligned} \tag{38}$$

with

$$\begin{aligned} H_{MM'}(u, \alpha_M) &= \delta_{MM'}\alpha_M (\sinh u)^{2\lambda - 3} (\cosh u)^{\tau_M + \tau_{M'} + 2M + 2M' - 2\lambda + 3} \\ &\quad + \frac{1}{\gamma^2} \sum_{m=0}^{M+M'} \frac{e_m^{(MM')} (\cosh u)^{2m}}{\tau_M + \tau_{M'} + 2M + 2M' - 2\lambda + 3 - 2m} \\ &\quad \times \left[1 - \gamma^2 + \left(\gamma^2 + \frac{3 - 2\lambda}{\tau_M + \tau_{M'} + 2M + 2M' - 2m} \right) \right. \\ &\quad \times F\left(\frac{1}{2}(\tau_M + \tau_{M'}) + M + M' - \lambda + \frac{3}{2} - m, 1; \right. \\ &\quad \left.\left. \frac{1}{2}(\tau_M + \tau_{M'}) + M + M' + 1 - m; \frac{1}{\cosh^2 u} \right) \right] \end{aligned} \tag{39}$$

and

$$e_m^{(MM')} \equiv \sum_{j=\max(0, m-M')}^{\min(m, M)} b_j^{(M)} b_{m-j}^{(M')}. \tag{40}$$

With these, the final form of the new potential (containing also the extra parameter α_M) becomes

$$\begin{aligned} V_2(r) &= V_0(r) + 2\frac{\gamma^2 + \sinh^2 u}{\sinh^2 u} \left[\frac{(q_M(\cosh u))^2}{H_{MM}(u, \alpha_M)} \right]^2 - \frac{2\gamma^2 (q_M(\cosh u))^2}{H_{MM}(u, \alpha_M)} \\ &\quad \times \left[-\frac{\cosh^2 u}{\gamma^2 + \sinh^2 u} + 2\tau_M + 4M + 1 + \frac{2\lambda - 2}{\sinh^2 u} - 2\cosh u \frac{q'_M(\cosh u)}{q_M(\cosh u)} \right]. \end{aligned} \tag{41}$$

In the $\gamma \to 1$ limit $V_2(r)$ reduces to the corresponding expression for the generalized Pöschl–Teller potential [17].

The new bound-state wavefunctions can be expressed in a similar manner, with the difference that in (6) another integral containing the f-type solution and a physical wavefunction together has to be evaluated. The resulting formula is

$$\begin{aligned} \psi_2^{(n)}(r) &= (\gamma^2 + \sinh^2 u)^{\frac{1}{4}} (\sinh u)^{\lambda} (\cosh u)^{-\mu_n - 2n - \lambda - \frac{1}{2}} \\ &\quad \times \left[p_n(\cosh u) - \coth^2 u \frac{L_{Mn}(u)}{H_{MM}(u, \alpha_M)} q_M(\cosh u) \right] \end{aligned} \tag{42}$$

where

$$L_{Mn}(u) = \frac{1}{\gamma^2} \sum_{m=0}^{n+M} \frac{g_m^{(Mn)} (\cosh u)^{2m}}{\tau_M + \mu_n + 2M + 2n - 2m} \times \left[1 + \frac{\gamma^2 - 1}{\cosh^2 u} \frac{\tau_M + \mu_n + 2M + 2n - 2m}{\tau_M + \mu_n + 2M + 2n + 2 - 2m} \right] \tag{43}$$

with

$$g_m^{(Mn)} \equiv \sum_{j=\max(0,m-n)}^{\min(m,M)} b_j^{(M)} c_{m-j}^{(n)}. \tag{44}$$

Using equations (6), (35) and (38) the normalized wavefunction of the new bound state can be written as

$$\psi_2^{(M)}(r) = \alpha_M^{1/2} (\gamma^2 + \sinh^2 u)^{\frac{1}{4}} (\sinh u)^{\lambda-2} (\cosh u)^{\tau_M + 2M - \lambda + \frac{3}{2}} \frac{q_M(\cosh u)}{H_{MM}(u, \alpha_M)}. \tag{45}$$

Figure 2 presents a potential of the type (41) obtained by inserting a new bound state between the two lowest states of the reference potential $V_0(r)$. This new bound state is labelled by 'a', while other bound states keep the same label. The potential presents a quite complicated structure, with several wells. This structure is strongly modified when other energies or parameters are chosen. Similarly to the phase-equivalent removal of a bound state, the generalized Levinson theorem (22) manifests itself by a change in the strength of the r^{-2}-type singularity at the origin. However, in this case λ is formally decreased with two units, and not increased as in the case of eliminating a state. The wavefunctions of $V_2(r)$ are also displayed in figure 2. The wavefunction with zero node and that of the new bound state have a complicated structure, corresponding to the potential shape.

With equation (7), m bound states can also be added at selected energies. This leads to more phase-equivalent potentials available analytically, provided that the condition $\lambda \geqslant 2m+1$ is satisfied. Here also, we focus on the $m = 2$ case for simplicity. The potential will depend on two energies related to τ_{M_1} and τ_{M_2} chosen among the values given by (34), and on two arbitrary parameters α_{M_1} and α_{M_2}. Its analytical expression reads

$$V_4(r) = V_0(r) + 2\frac{\mathrm{d}}{\mathrm{d}r} \frac{(f_0^{(M_1)})^2 J_{M_2M_2} + (f_0^{(M_2)})^2 J_{M_1M_1} - 2 f_0^{(M_1)} f_0^{(M_2)} J_{M_1M_2}}{J_{M_1M_1} J_{M_2M_2} - (J_{M_1M_2})^2} \tag{46}$$

where r is again implied. The situation is similar to that discussed after equation (31), and the wavefunctions of the different bound states at energies E_n and E_{M_i} can again be written explicitly in terms of the $J_{M_iM_j}$ and J_{M_in} and of the functions $\psi_0^{(n)}$ and $f_0^{(M_i)}$.

The other possible choice for reducing the hypergeometric function in (32) and (16) to polynomial form, $a = -M$, yields $\frac{\mathrm{i}k}{\gamma^2} + \lambda - 2 - \sigma(k) - 2M = 0$, which puts more severe constraints on the possible M values than the corresponding formulae for $b = -M$. It turns out, for example, that with $a = -M$ new bound states can be introduced for potentials that originally did not support any bound states.

3.4. Unchanged spectrum

For the sake of completeness we finally present formulae for the third kind of operation appearing in table 1, i.e. the pair of transformations that first eliminates a state and then reintroduces it at the same energy, leaving the spectrum unchanged, introducing only a new parameter.

Similarly to the phase-equivalent removal of states here also the physical wavefunctions ψ_0 have to be applied. Making use of the normalizability of the bound-state wavefunctions the integral appearing in (5) can be expressed in terms of the $I_{Nn}(r)$s of (26):

$$\frac{\alpha}{1-\alpha}+\int_r^\infty (\psi_0^{(N)}(t))^2\,\mathrm{d}t=\frac{1}{1-\alpha}-\int_0^r (\psi_0^{(N)}(t))^2\,\mathrm{d}t\equiv\frac{1}{1-\alpha}-I_{NN}(r). \tag{47}$$

The new potential is then

$$V_2(r)=V_0(r)+\frac{4\psi_0^{(N)}(r)(\psi_0^{(N)}(r))'}{\frac{1}{1-\alpha}-I_{NN}(r)}+\frac{2(\psi_0^{(N)}(r))^4}{\left(\frac{1}{1-\alpha}-I_{NN}(r)\right)^2}. \tag{48}$$

Its explicit expression is given by (29) with the replacement

$$G_{NN}\to G_{NN}-(1-\alpha)^{-1}(\sinh u)^{-2\lambda-1}(\cosh u)^{2\mu_N+4N+2\lambda-1}. \tag{49}$$

According to (6) the new bound-state solutions take the form

$$\psi_2^{(n)}(r)=[1+(\alpha-1)\delta_{Nn}]^{-1/2}\left(\psi_0^{(n)}(r)-\psi_0^{(N)}(r)\frac{\delta_{Nn}-I_{Nn}(r)}{\frac{1}{1-\alpha}-I_{NN}(r)}\right). \tag{50}$$

Their explicit expression is given by (30) with replacements similar to (49). Just as in the case of inserting a new bound state, a parameter appeared in the formulae. The $\alpha\to 1$ limit corresponds to the trivial $V_2(r)=V_0(r)$ transformation, while $\alpha\to\infty$ gives the removal case. Even more parameters can be introduced by iterating this type of transformation, similarly, for example to (46).

Figure 3 presents a potential of the type (48) obtained by modifying the normalization constant (used in the sense of inverse scattering theory [7]) of the first excited state of $V_0(r)$. The generalized Levinson theorem is not modified this time, neither is the potential singularity, because the number of bound states is unchanged. The wavefunctions are also given in figure 3. Let us note that the asymptotic behaviour of these wavefunctions is not modified by the transformation, except for the first excited state. This feature is related to phase equivalence [11], and is also true in the removal and addition cases, as can be seen in figures 1 and 2. Moreover, in the unchanged spectrum case, the behaviour of the wavefunctions at the origin is also unaffected, except for the modified state. This can be seen in equation (50) where for $n\neq N$ the limit for $r\to 0$ does not depend on α.

4. Summary and conclusions

We applied the abstract formalism of supersymmetric quantum mechanics to the specific case of the generalized Ginocchio potential and explicitly derived closed algebraic expressions for its phase-equivalent partners. The spectrum of the resulting potential was identical with the generalized Ginocchio potential, except possibly for certain bound states which were either removed from the spectrum or were added to it. For the generalized Ginocchio potential, analytical expressions of the Jost function and of the S-matrix are available. Because of phase-equivalence, the S-matrix is not modified by the transformations described here. The Jost function is multiplied by a rational function of the wave number when a bound state is added or removed, and is unchanged when the spectrum is unchanged.

Our work is the first example for deriving phase-equivalent partners of a potential outside the shape-invariant class using the formalism of SUSYQM. Another novelty is that we also gave closed analytical expressions for the bound-state wavefunctions of the new potential. We note that our results are directly applicable to the generalized Pöschl–Teller potential by taking the $\gamma=1$ substitution.

One type of transformations was the removal of an arbitrary bound state from the spectrum of the generalized Ginocchio potential. The definite integrals appearing in the formulae could be calculated analytically in any case here. As another example we added a new bound state to the spectrum of the generalized Ginocchio potential. Then the integrand contained hypergeometric functions, and the integrals could be evaluated only in special cases, which meant that the new bound state could be inserted only at specific energies. This procedure requires an r^{-2}-like repulsive singularity of the original potential, therefore it is generalizable only to potentials that have this feature. This forbids a similar treatment of a number of potentials (Morse, Hulthén, Rosen–Morse, etc). We also discussed transformations which do not change the spectrum, but introduce one (or more) parameter in the potential. This type of transformation is known from inverse scattering theory and is applicable to other types of potentials as well. We briefly outlined how the three transformation types can be iterated.

Similarly to other fully analytical transformations, our results might be helpful in testing numerical methods in situations that might be problematic in terms of numerical techniques. This is the case, for example, for certain types of complex potentials [19, 20]: our formulae are applicable to complex Ginocchio potentials without any major modification. The particular case of the Ginocchio potential offers analytical results for a potential with rather flexible shape, which can be considered as a reasonable approximation of realistic potentials used in nuclear physics, for example. Similar treatment of further rather general Natanzon-class potentials also seems possible. These include examples close to potential shapes used in atomic, molecular, solid state physics and other areas.

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